GROWTH OF BALLS IN THE UNIVERSAL COVER OF SURFACES AND GRAPHS.

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ABSTRACT. In this paper, we prove uniform lower bounds on the volume growth of balls in the universal covers of Riemannian surfaces and graphs. More precisely, there exists a constant $\delta > 0$ such that if (M, hyp) is a closed hyperbolic surface and h another metric on M with $\mathrm{Area}(M, h) \leq \delta \, \mathrm{Area}(M, hyp)$ then for every radius $R \geq 1$ the universal cover of (M, h) contains an R-ball with area at least the area of an R-ball in the hyperbolic plane. This positively answers a question of L. Guth for surfaces. We also prove an analog theorem for graphs.

1. Introduction

Let $(\widetilde{M}, \widetilde{h})$ be the universal cover of a closed Riemannian manifold (M, h). We consider the function

$$V_{(\widetilde{M},\widetilde{h})}(R) := \sup_{\tilde{x}\in\widetilde{M}} \mathrm{Vol}\, B_{\widetilde{h}}(\tilde{x}, R).$$

The function $V(R)$ is the largest volume of any ball of radius R in $(\widetilde{M}, \widetilde{h})$. Since it is possible to construct examples of Riemannian manifolds where the volume of some balls of radius R in the universal cover is arbitrary small, it is interesting to know whether there is at least one ball of radius R in the universal cover with a large volume. If the curvature of the metric h is bounded above by a negative constant then the Bishop-Gunther-Gromov inequality gives us an exponential lower bound on the volume of all balls in the universal cover $\widetilde{M}$. So in particular we have an estimate of the function V. In this paper, we are interested in finding curvature-free exponential lower bounds for V. We replace the local assumption, namely a curvature bound, by a topological assumption and a condition on the volume of (M, h). What is believed is that if the topology of M is complicated then the function V is large (see [6] and [8] for more details).

Before going further, we would like to point out that the function $V_{(\widetilde{M},\widetilde{h})}$ is related to the volume entropy of (M, h). The volume entropy of (M, h) is defined as

$$\mathrm{Ent}(M, h) = \lim_{R\to+\infty} \log(Vol(B_{\widetilde{h}}(\tilde{x}, R))).$$

Since M is compact, the limit exists and does not depend on the point $\tilde{x}$ (see [13]). The volume entropy is a way of describing the asymptotic behavior

of the volumes of balls in the universal cover of a given Riemannian manifold.

An example of a manifold with "complicated topology" is a manifold of hyperbolic type, i.e., a manifold which admits a hyperbolic Riemannian metric. Let (M^n, hyp) be a closed hyperbolic manifold. The volume of a ball in the hyperbolic space $\mathbb{H}^n$, i.e., the universal cover of (M^n, hyp), is independent of the center of the ball. Thus $V_{\mathbb{H}^n}(R)$ is just the volume of any ball of radius R in the hyperbolic n-space, which can be explicitly calculated. In particular, when $n = 2$, for every $R > 0$ we have

$$V_{\mathbb{H}^2}(R) = 2\pi(\cosh(R) - 1). \tag{1.1}$$

So there exists a constant c such that

$$V_{\mathbb{H}^2}(R) \sim ce^R,$$

when R goes to infinity.

Now let h be another metric on M with $\mathrm{Vol}(M, h) \leq \mathrm{Vol}(M, hyp)$. Does the balls in $(\widetilde{M}, \widetilde{h})$ also grow exponentially like in the hyperbolic case? There exist two fundamental theorems in this direction. The first theorem is due to G. Besson, G. Courtois, S. Gallot [1] and also to A. Katok [9] for the dimension $n = 2$. The authors proved that if M is a closed connected Riemannian manifold that carries a rank one locally symmetric metric h_0, then for every Riemannian metric h such that $\mathrm{Vol}(M, h) = \mathrm{Vol}(M, h_0)$, the inequality $\mathrm{Ent}(M, h) \geq \mathrm{Ent}(M, h_0)$ holds. In our language their theorem can be expressed as follows.

Theorem 1.1 (see [1], [9]). *Let (M^n, hyp) be a closed hyperbolic manifold, and let h be another metric on M with $\mathrm{Vol}(M, h) < \mathrm{Vol}(M, hyp)$. Then there is some constant R_0 (depending on the metric h) such that for every radius $R > R_0$, the following inequality holds:*

$$V_{(\widetilde{M}, \widetilde{h})}(R) > V_{\mathbb{H}^n}(R).$$

It would be interesting to know the value of R_0 in Theorem 1.1 since we are looking for a lower bound on the function $V_{(\widetilde{M}, \widetilde{h})}$ for every $R \geq 0$.
The second fundamental theorem can be seen as a first step toward estimating R_0 but with a stronger hypothesis.

Theorem 1.2 (Guth, [8]). *For every dimension n, there is a number $\delta(n) > 0$ such that if (M^n, hyp) is a closed hyperbolic n-manifold and h is another metric on M with $\mathrm{Vol}(M, h) < \delta(n)\,\mathrm{Vol}(M, hyp)$, then the following inequality holds*

$$V_{(\widetilde{M}, \widetilde{h})}(1) > V_{\mathbb{H}^n}(1).$$

The method presented in [8] can be modified to give a similar estimate for balls of radius R. For each R, there is a constant $\delta(n, R) > 0$ such

that if $\mathrm{Vol}(M,g) < \delta(n,R)\,\mathrm{Vol}(M,hyp)$ then $V_{(\tilde{M},\tilde{g})}(R) > V_{\mathbb{H}^n}(R)$. As R goes to infinity, the constant $\delta(n,R)$ falls off exponentially or faster so this method become less effective, whereas the methods in [1] are only effective asymptotically for very large R. This led L. Guth to ask if we can get a uniform estimate for $R \geq 1$. In other words, the question is: does there exist a positive constant $\delta(n)$ such that $\mathrm{Vol}(M,g) < \delta(n)\,\mathrm{Vol}(M,hyp)$ implies $V_{(\tilde{M},\tilde{g})}(R) > V_{\mathbb{H}^n}(R)$ for all $R \geq 1$?

Here we positively answer Guth's question for the dimension $n = 2$.

Theorem A. *There exists a positive constant δ such that if (M, hyp) is a closed hyperbolic surface and h is another metric on M with* $\mathrm{Area}(M,h) \leq \delta\,\mathrm{Area}(M,hyp)$*, then for any radius $R \geq 1$,*

$$V_{(\widetilde{M},\widetilde{h})}(R) \geq V_{\mathbb{H}^2}(R).$$

Our Theorem A will be deduced from the following more general theorem.

Theorem B. *There exists two small positive constants δ and c such that if (M, hyp) is a closed hyperbolic surface and h is another metric on M with* $\mathrm{Area}(M,h) \leq \delta\,\mathrm{Area}(M,hyp)$*, then for any radius $R \geq 0$,*

$$V_{(\widetilde{M},\widetilde{h})}(R) \geq V_{\mathbb{H}^2}(cR).$$

We can extend the notion of entropy from Riemannian manifolds to metric graphs. Let (Γ, h) be a metric graph and denote by $(\widetilde{\Gamma}, \widetilde{h})$ its universal cover. Fix a point v of Γ and a lift $\tilde{v}$ of this point in $\tilde{\Gamma}$. The volume entropy of (Γ, d) is defined as

$$\mathrm{Ent}(\Gamma,h) = \lim_{R\to\infty} \frac{\log(\mathrm{length}(B_{\widetilde{h}}(\tilde{v},R)))}{R}.$$

Since Γ is compact, the limit exists and does not depend on the point $\tilde{v}$ (see [13]).

Definition 1.3. Let (Γ, h) be a metric graph and denote by $(\widetilde{\Gamma}, \widetilde{h})$ its universal cover. We define the function

$$V'(R) := \sup_{\tilde{v}\in\widetilde{\Gamma}} \mathrm{length}(B_{\widetilde{h}}(\tilde{v},R)),$$

where $B_{\widetilde{h}}(\tilde{v},R)$ is a ball of radius R centered at the point $\tilde{v}$ of $\widetilde{\Gamma}$.

A regular graph is the analog of a Riemannian manifold carrying a locally symmetric metric. For every positive integer $b \geq 2$, we denote by Γ_b the connected trivalent graph of first Betti number b and by h_b the metric on Γ_b for which all the edges have length 1. In [10] (see also [11]), the authors proved a theorem for graphs analog to the G. Besson, G. Courtois and S. Gallot theorem for manifolds. They showed that for every integer

$b \geq 2$ and every connected metric graph (Γ, h) of first Betti number b such that $\operatorname{length}(\Gamma, h) = \operatorname{length}(\Gamma_b, h_b)$, we have $\operatorname{Ent}(\Gamma, h) \geq \operatorname{Ent}(\Gamma_b, h_b)$. In our language, their theorem can be stated as follows.

Theorem 1.4 ([10],[11])**.** *Let (Γ, h) be a connected metric graph of first Betti number $b \geq 2$ Such that $\operatorname{length}(\Gamma, h) < \operatorname{length}(\Gamma_b, h_b)$. Then there exists some constant R'_0 (depending on the metric h) such that for every radius $R > R'_0$ the following inequality holds*

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

In view of Theorems 1.2 and 1.4, one can ask the following question: does there exist a universal constant $c > 0$ such that if $\operatorname{length}(\Gamma, h) < c \operatorname{length}(\Gamma_b, h_b)$, then for all $R \geq 0$

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R)?$$

We give a partial answer to this question.

Theorem C. *Fix $\lambda \in (0, \frac{1}{3})$. Let (Γ, h) be a connected metric graph of first Betti number $b \geq 2$ such that*

$$\operatorname{length}(\Gamma, h) \leq \lambda \operatorname{length}(\Gamma_b, h_b).$$

Then there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that for any $R \geq 0$, we have

$$\operatorname{length} B_{\widetilde{h}}(\tilde{u}, R) \geq (1 - 3\lambda) V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq (1 - 3\lambda) V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

We sketch an outline of the main idea of the proof of Theorem B. Fix $R \geq 0$ and denote by g the genus of M. First, we show that we can suppose that the systole $\operatorname{sys}(M, h)$ of (M, h) is at least $max\{2R, 1/2\}$. This lower bound on the systole and the upper bound on the area of the surface in terms of the genus permit us to show the existence of an embedded minimal graph Γ in M which captures the topology of the surface (*cf.* Definition 5.1 and Definition 5.5) and satisfies the hypothesis of Theorem C. Therefore, there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that for all radii $r \in (0, R)$, the length of the ball $B_{\widetilde{\Gamma}}(r)$ in $\widetilde{\Gamma}$ centered at $\tilde{u}$ and of radius r is large. Since $R \leq \frac{1}{2} \operatorname{sys}(\Gamma, h)$, the length of the projection $B_\Gamma(r)$ of $B_{\widetilde{\Gamma}}(r)$ in Γ is also large. Let $B_M(r)$ be the ball of radius r in M with the same center as $B_\Gamma(r)$. For all radii $r \leq R$, the boundary of $B_M(r)$ is at least as long as the graph $\Gamma \cap B_M(r)$, for otherwise we could construct another graph Γ' which captures the topology of M and is shorter than Γ. This would contradict the minimality of Γ. Since the graph $\Gamma \cap B_M(r)$ contains $B_\Gamma(r)$, we derive that the length of $\partial B_M(r)$ is large. By the coarea formula, we conclude that the area of B_R is also large.

This paper is organized as follows. In Section 2, we recall the basic material of graphs we need in this paper. In Section 3, we prove a special case of Theorem C. In Section 4, we prove Theorem C in the general case. In Section 5, we show the existence of graphs that captures the topology of closed orientable Riemannian surfaces. In Section 6, we extend the notion of the height function originally defined by Gromov for surfaces, then we show a relation between the height and the area of balls. In Section 7, we establish the existence of ε-regular metrics. In Section 8, we define short minimal graphs on surfaces that capture the topology and we study their properties. At the end of this section, we show how to control their length in terms of the genus of the surface. In Section 9, we give the proof of the main theorems A and B.

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2. Preliminaries

By a graph Γ we mean a finite one-dimensional CW-complexe (multiple edges and loops are allowed). It is also useful to see Γ as a pair of sets (V, E) where V is a set of vertices and E the set of edges, which are 2-element subsets of V. Two vertices of a graph are called *adjacent* if there is an edge linking them. An edge and a vertex are called *incident* if the vertex is an endpoint of the edge. The *degree* (also known as valence) of a vertex v, denoted by $\deg(v)$, is the number of edges incident to it, where the loops are counted twice. We say that a graph Γ is *k-regular* if the degree of any vertex is k. In particular, a 3-regular graph is called *trivalent*. The minimal degree of a graph Γ is the minimum of the degrees of the vertices. It will be denoted by $\mathrm{Mindeg}(\Gamma)$. A graph Γ with $\mathrm{Mindeg}(\Gamma) \geq 3$ is called at least trivalent. For a graph Γ, we always denote by $E(\Gamma)$ the set of its edges and by $V(\Gamma)$ the set of its vertices. The first Betti number of a graph Γ can be computed as follows:

$$b(\Gamma) = e - v + n, \tag{2.1}$$

where e, v and n are respectively the number of edges, vertices and connected components of Γ.

The degree sum formula states that, given a graph Γ, we have that

$$\sum_v \deg(v) = 2e, \tag{2.2}$$

where the summation is over all vertices v of Γ.

For an at least trivalent connected graph Γ with first Betti number b, we have that $2e \geq 3v$ by (2.2). Combined with (2.1), we get $e \leq 3b - 3$. That means that the number of edges of Γ is bounded in terms of its first Betti number b. Also it is not hard to see from (2.1) and (2.2) that every

connected graph of first Betti number $b \geq 2$ has at least one vertex of degree at least 3.

Let Γ be a connected graph, v_0 and v_1 be two vertices of Γ. A path P from v_0 to v_1 is a sequence of directed edges that links v_0 to v_1. The vertex v_0 is called the start point of P and v_1 the endpoint. If $v_0 = v_1$ then P is said to be closed, otherwise P is open. A simple path is a path with no self intersections. A simple closed path is often called a cycle.

A metric graph (Γ, h) is a graph endowed with a metric h such that (Γ, h) is a length space. The length of a subgraph of Γ is its one-dimensional Hausdorff measure. For more details on graphs we refer the reader to [4].

Throughout this paper if R is a real number then $[R]$ is the integral part of R.

For the connected trivalent metric graph (Γ_b, h_b) of first Betti number $b \geq 2$ where edges are of unit length, the following holds:

- $\mathrm{length}(\Gamma_b, h_b) = 3b - 3.$ (2.3)
- The universal cover $\widetilde{\Gamma}_b$ is isometric to the trivalent infinite tree. In particular, $\widetilde{\Gamma}_b$ is independent of b. So for every $b' \geq 2$ we have
$$V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R) = V'_{(\widetilde{\Gamma}_{b'}, \widetilde{h}_{b'})}(R).$$
- For every $R \geq 0$ and every vertex $\tilde{v}$ of $(\widetilde{\Gamma}_b, \widetilde{h}_b)$, we have
$$\begin{aligned} \mathrm{length}(B_{\widetilde{h}_b}(\tilde{v}, R)) &= 3 \sum_{n=0}^{[R]-1} 2^n + 3(R - [R])2^{[R]} \\ &= 3(2^{[R]} - 1) + 3(R - [R])2^{[R]} \\ &\geq \sinh(R \ln 2). \end{aligned} \tag{2.4}$$
Therefore, $V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R) \geq \sinh(R \ln 2)$.
In particular, one should notice that the volume of the ball $B_{\widetilde{h}_b}(\tilde{v}, R)$ is independent from the vertex $\tilde{v}$ and from the first Betti number b. It only depends on R.

3. Baby theorem C

In this section, we prove Theorem C with an additional bound on the lengths of the edges of Γ and on the minimal degree of Γ (*cf.* Section 2).

Proposition 3.1. *Let c and C' be two positive constants with $c \leq C'$. Let (Γ, h) be a connected, at least trivalent metric graph of first Betti number*

$b \geq 2$ such that the edges of Γ are of length at most c. Then there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that for any $R \geq 0$, we have

$$\text{length}\, B_{\widetilde{h}}(\tilde{u}, (C' + c)R) \geq C' V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma}, \widetilde{h})}((C' + c)R) \geq C' V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R).$$

Proof. Let $\mathcal{T}$ be a connected trivalent infinite subgraph of $\widetilde{\Gamma}$. We will construct a connected trivalent infinite subgraph $\mathcal{T}'$ of $\mathcal{T}$ for which there exists an homeomorphism $f : \widetilde{\Gamma}_b \to \mathcal{T}'$ that satisfies the following:
For every pair of vertices x, y of $\widetilde{\Gamma}_b$, we have

$$C' d(x, y) \leq d(f(x), f(y)) \leq (C' + c) d(x, y). \tag{3.1}$$

For the sake of clarification, we will do this construction step by step.

Step 1: Start by fixing a vertex v_0 in $\mathcal{T}$. Let e_{1v_0} be one of the three edges of $\mathcal{T}$ incident to v_0 and denote by v_1 its second endpoint. Again let e_{1v_1} be one of the other two edges of $\mathcal{T}$ incident to v_1 and denote by v_2 its second endpoint. The path $e_{1v_0}e_{1v_1}$ is simple and open. We continue doing this by induction and we denote by v_k the first vertex where the length of the path $e_{1v_0}...e_{1v_k}$ is at least C'. The graph $\mathcal{T}$ contains no nontrivial cycles since it is a tree. That means that the path $p_1 = e_{1v_0}...e_{1v_k}$ is simple and open. Furthermore, the length of p_1 is between C' and $C' + c$. Now take the second edge e_{2v_0} of $\mathcal{T}$ incident to v_0 and restart the process of Step 1. This give us another simple open path p_2. Again, since $\mathcal{T}$ contains no nontrivial cycles the intersection $p_1 \cap p_2$ is the vertex v_0. Also restart the process with the third edge of $\mathcal{T}$ incident to v_0 to get the third path p_3.

Step 2: The tree $X = p_1 \cup p_2 \cup p_3$ has three leaves. For each leaf x_i of X there are two edges of $\mathcal{T}$ incident to it other than the edge that is already in X. So by restarting the process of Step 1, we construct two paths of length at least C' with start point x_i. By induction, we keep doing what we did before to finally get the subgraph $\mathcal{T}'$. In what follows each path p_i of the subgraph $\mathcal{T}'$ will be seen as an edge of the same length of p_i. That means $\mathcal{T}'$ can be seen as a connected infinite trivalent subgraph of $\mathcal{T}$ where the length of any edge of $\mathcal{T}'$ is between C' and $C' + c$. The graphs $\widetilde{\Gamma}_b$ and $\mathcal{T}'$ are two infinite trivalent trees so there exists an homeomorphism $f : \widetilde{\Gamma}_b \to \mathcal{T}'$ that sends every edge of $\widetilde{\Gamma}_b$ to an edge of $\mathcal{T}'$.

Now we prove that the map f satisfies (3.1). Without loss of generality, we will prove our claim when x and y are the endpoints of the same edge e_{xy} in $\widetilde{\Gamma}_b$, that is, $d(x, y) = 1$. By construction of the map f, the length of the image of an edge of $\widetilde{\Gamma}_b$ is between C' and $C' + c$. So clearly

$$C' d(x, y) \leq d(f(x), f(y)) \leq (C' + c) d(x, y).$$

Now let $\tilde{u}$ be a vertex of $\mathcal{T}'$ and denote by w its inverse image in $\widetilde{\Gamma}_b$. By (3.1), we have

$$\begin{aligned} C' \operatorname{length} B_{\widetilde{h}_b}(w,R) &\leq \operatorname{length}(f(B_{\widetilde{h}_b}(w,R))) \\ &\leq \operatorname{length}(B_{\widetilde{h}}(\tilde{u},(C'+c)R)), \end{aligned}$$

Hence the proposition. $\square$

4. Proof of theorem C

In this section, we prove Theorem C. As a preliminary, let us examine how the function V' changes with scaling. Let (Γ,h) be a metric graph and $h'=\mu h$ with $\mu>0$ then

- $\operatorname{length}(\Gamma,h')=\mu \operatorname{length}(\Gamma,h)$;
- $V'_{(\widetilde{\Gamma},\widetilde{h'})}(\mu R)=\mu V'_{(\widetilde{\Gamma},\widetilde{h})}(R)$.

Definition 4.1. Let Γ be a connected metric graph of first Betti number at least two. If v is a vertex of Γ of degree two then by the sentence "*ignore the vertex v*" we mean delete the two edges e_1 and e_2 of Γ incident to v and replace them by an edge of length $\operatorname{length}(e_1)+\operatorname{length}(e_2)$ that links the other two vertices of e_1 and e_2.

Lemma 4.2. *Let (Γ,h) be a connected metric graph of first Betti number $b\geq 2$. There exists a metric graph (Γ',h') with first Betti number $b'=b$ that satisfies the following.*

- *Γ' is at least trivalent;*
- *$\operatorname{length}(\Gamma',h')\leq \operatorname{length}(\Gamma,h)$;*
- *For all $R\geq 0$,*

$$V'_{(\widetilde{\Gamma'},\widetilde{h'})}(R)\leq V'_{(\widetilde{\Gamma},\widetilde{h})}(R).$$

Proof. First we remove every vertex of Γ of degree one along with the edge incident to it and denote by Γ_1 the resulting connected graph. We apply the same process to Γ_1. That means we remove every vertex of Γ_1 of degree one along with the edge incident to it and we denote by Γ_2 the resulting connected graph . By induction, let Γ_k be the last connected graph where no vertex of degree one left. The graph Γ_k is of first Betti number b and of length less or equal to the length of Γ. We keep denoting by h the restriction of the metric h to Γ_k. The universal cover $\widetilde{\Gamma}_k$ is isometrically embedded into $\widetilde{\Gamma}$ so

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R)\geq V'_{(\widetilde{\Gamma}_k,\widetilde{h})}(R).$$

Second, we ignore every vertex of Γ_k of degree two (*cf.* Definiton 4.1). The resulting graph Γ' is connected of first Betti number b and of the same length as Γ_k. The universal cover $\widetilde{\Gamma}'$ agrees with $\widetilde{\Gamma}_k$ so

$$V'_{(\widetilde{\Gamma}',\widetilde{h})}(R)=V'_{(\widetilde{\Gamma}_k,\widetilde{h})}(R).$$

$\square$

In order to prove Theorem C, it is convenient here to reformulate it. Given $\lambda \in (0, \frac{1}{3})$, let c and C' be two positive constants such that $c \leq C'$ and $\lambda = \frac{c}{3(C'+c)}$. So a reformulated version of Theorem C is the following.

Theorem 4.3. *Let (Γ, h) be a connected metric graph of first Betti number $b \geq 2$. Let C' and c be two positive constants with $c \leq C'$. Suppose that*

$$\mathrm{length}(\Gamma, h) \leq \frac{c}{3(C'+c)} \mathrm{length}(\Gamma_b, h_b).$$

Then there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that for any $R \geq 0$, we have

$$\mathrm{length}\, B_{\widetilde{h}}(\tilde{u}, R) \geq \frac{C'}{C'+c} V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma}, \widetilde{h})}(R) \geq \frac{C'}{C'+c} V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R).$$

Proof. By scaling, we will prove the following. Suppose that

$$\mathrm{length}(\Gamma, h) \leq \frac{c}{3} \mathrm{length}(\Gamma_b, h_b) = c(b-1).$$

Then there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that for any $R \geq 0$, we have

$$\mathrm{length}\, B_{\widetilde{h}}(\tilde{u}, (C'+c)R) \geq C' V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma}, \widetilde{h})}((C'+c)R) \geq C' V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R).$$

First notice that by Lemma 4.2, we can suppose that Γ is at least trivalent. We proceed by induction on the first Betti number of Γ. For $b = 2$, we have

$$\max_{e \in E} \mathrm{length}(e) < \mathrm{length}(\Gamma, h) \leq c(2-1) = c.$$

By Proposition 3.1, the result follows in this case.
Suppose the result holds for $b = n$ and let us show that it also for $b = n+1$. Let (Γ, h) be a connected metric graph of first Betti number $b = n+1$. If Γ contains no edge of length greater than c then the result follows from Proposition 3.1. Thus we suppose the opposite here and remove an edge w of Γ of length greater than c. There are two cases to consider.

Case 1: The edge w is non-separating in Γ. In this case, the resulting graph Γ' is connected and of first Betti number $b' = n$. Furthermore, we have

$$\mathrm{length}(\Gamma') \leq \mathrm{length}(\Gamma) - c \leq c(b'-1).$$

The universal cover $\widetilde{\Gamma}'$ is isometrically embedded into $\widetilde{\Gamma}$. So for every vertex $\tilde{v}$ in $\widetilde{\Gamma}'$ and every $R > 0$, we have

$$\mathrm{length}(B_{(\widetilde{\Gamma},\widetilde{h})}(\tilde{v}, R)) \geq \mathrm{length}(B_{(\widetilde{\Gamma}',\widetilde{h})}(\tilde{v}, R)).$$

In particular, we have

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}',\widetilde{h})}(R).$$

On the other hand, by the hypothesis of the induction, we know that there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}'$ such that

$$\mathrm{length}(B_{(\widetilde{\Gamma}',\widetilde{h})}(\tilde{u}, R)) \geq V'_{(\widetilde{\Gamma}_n,\widetilde{h}_n)}(R) = V'_{(\widetilde{\Gamma}_{n+1},\widetilde{h}_{n+1})}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma}',\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_{n+1},\widetilde{h}_{n+1})}(R).$$

This finishes the proof in this case.

Case 2: The edge w is separating in Γ. Thus, it splits the graph Γ into two connected graphs Γ' and Γ'' of first Betti number b' and b''. We claim that $\mathrm{length}(\Gamma') \leq c(b' - 1)$ or $\mathrm{length}(\Gamma'') \leq c(b'' - 1)$. Indeed, suppose the opposite then

$$\mathrm{length}(\Gamma') + \mathrm{length}(\Gamma'') > c(b - 2).$$

On the other hand we have

$$\mathrm{length}(\Gamma') + \mathrm{length}(\Gamma'') + c < \mathrm{length}(\Gamma) \leq c(b - 1).$$

Hence a contradiction. So the claim is proved.
Without loss of generality, suppose that Γ' satisfies $\mathrm{length}(\Gamma') \leq c(b' - 1)$. Clearly $b' \geq 2$, otherwise the length of Γ' would vanish. By induction, we now there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}'$ such that

$$\mathrm{length}(B_{(\widetilde{\Gamma}',\widetilde{h})}(\tilde{u}, R)) \geq V'_{(\widetilde{\Gamma}_{b'},\widetilde{h}_{b'})}(R) = V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma}',\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

Recall that the universal cover $\widetilde{\Gamma}'$ is isometrically embedded into $\widetilde{\Gamma}$. So for every vertex $\tilde{v}$ in $\widetilde{\Gamma}'$ and every $R > 0$, we have

$$\mathrm{length}(B_{(\widetilde{\Gamma},\widetilde{h})}(\tilde{v}, R)) \geq \mathrm{length}(B_{(\widetilde{\Gamma}',\widetilde{h})}(\tilde{v}, R)).$$

In particular, we have

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}',\widetilde{h})}(R).$$

This finishes the proof in this case too.

$\square$

5. Capturing the topology of surfaces

In this section, we show that on every closed orientable Riemannian surface M there exist an embedded graph that captures its topology.

Definition 5.1. Let (M, h) be a closed Riemannian surface of genus g. The image in M of an abstract graph by an embedding will be refered to as a graph in M. The metric h on M naturally induces a metric on a graph Γ in M. Despite the risk of confusion, we will also denote by h such a metric on Γ.

We say that a graph Γ in M *captures the topology* of M if the map induced by the inclusion $i_* : H_1(\Gamma, \mathbb{R}) \to H_1(M, \mathbb{R})$ is an epimorphism.

Lemma 5.2. *Let (M, h) be a closed orientable Riemannian surface. Let Γ be a connected graph in M that captures its topology and denote by $i : \Gamma \to M$ the inclusion map. Then there exists a connected subgraph Γ' of Γ such that the map i_* restricted to Γ' is an isomorphism. In particular the first Betti number of Γ' is $2g$.*

Proof. Let Γ' be a connected subgraph of Γ with minimal number of edges such that the restriction of i to Γ' still induces an epimorphism in real homology. Let α be a cycle of Γ' representing a nontrivial element of the kernel of i_*. Remove an edge e from α. The resulting graph Γ'' has fewer edges than Γ'. Let β be a cycle of Γ'. If e does not lie in β then the cycle $\gamma = \beta$ lies in Γ''. Otherwise, adding a suitable real multiple of α to β yields a new cycle γ lying in Γ''. In both cases, the cycle γ of Γ'' is sent to the same homology class as β by i_*. Thus, the restriction of i to Γ'' still induces an epimorphism in the real homology, which is absurd by definition of Γ'. □

In what follows a graph Γ in a Riemannian manifold (M, h) is automatically equipped with the metric h induced by the metric of M. So the length of Γ is its one-dimensional Hausdorff measure associated to the metric h.

Definition 5.3. Let (M, h) be a closed orientable Riemannian surface. We define

$$L(M, h) := \inf_{\Gamma} \operatorname{length}(\Gamma),$$

where the infimum is taken over all graphs Γ in M that capture its topology.

Lemma 5.4. *Let (M, h) be a closed orientable Riemannian surface of genus g. Then there exists a graph Γ in M that captures its topology with*

$$\operatorname{length}(\Gamma) = L(M, h).$$

Proof. By Lemma 5.2, we only need to consider the set of graphs in M that captures its topology with first Betti number $2g$ and such that i_* is an isomorphism. Furthermore, we only need to consider graphs that are at

least trivalent. Indeed, delete every vertex of Γ of degree one along with the edge incident to it. Denote by Γ_1 the resulting connected graph and apply to Γ_1 the same process. That means we delete every vertex of Γ_1 of degree one along with the edge incident to it and we denote by Γ_2 the resulting connected graph . By induction, let Γ_k be the last connected graph with no vertex of degree one. We then ignore all vertices of Γ_k of degree two (*cf.* Definition 4.1). Replacing every edge of Γ_k by a minimal representative of its fixed-endpoint homotopy class gives rise to a geodesic graph Γ'. By construction the connected geodesic graph Γ' is at least trivalent and of first Betti number $2g$. Thus, its number of edges is bounded in terms of g, *cf.* Section 2. Now the space of connected geodesic graphs of M capturing its topology with bounded length and a bounded number of edges is compact. The result follows. $\square$

Definition 5.5. Let (M, h) be a closed orientable Riemannian surface. If Γ is a graph that captures the topology of M with $\mathrm{length}(\Gamma) = L(M, h)$, then Γ is called a minimal graph in M.

6. Height function and area of balls.

In this section, we first recall the definition of the *height* function on surfaces defined by Gromov in [5] along with its relation to the area of balls. Then we extend this notion to make it suit our problem.

Let M be a closed Riemannian manifold. The systole at a point x in M, denoted by $\mathrm{sys}(M, x)$, is the length of the shortest non-contractible loop based at x. The systole of M, denoted by $\mathrm{sys}(M)$, is the length of the shortest non-contractible loop in M.

Definition 6.1. Let (M, h) be a closed Riemannian surface and γ be a non-contractible loop in M. We define the tension of γ as follows.

$$\mathrm{tens}(\gamma) = \mathrm{length}(\gamma) - \inf_{\beta \sim \gamma}(\mathrm{length}(\beta)),$$

where the infimum is taken over all closed curves β freely homotopic to γ.

We also define the height function H' on M as follows

$$H'(x) = \inf_{\gamma}(\mathrm{tens}(\gamma)),$$

where the infimum is taken over all non-contractible closed curves γ passing through x.

Proposition 6.2 (Gromov, [5] Proposition 5.1.B)**.** *Let (M, h) be a complete Riemannian surface and $x \in M$. Then*

$$\mathrm{Area}\, B(x, R) \geq \frac{1}{2}(2R - H'(x))^2,$$

for every R in the interval $[\frac{1}{2}H'(x), \frac{1}{2}\,\mathrm{sys}(M, x)]$.

Definition 6.3. Let (M, h) be a closed orientable Riemannian surface. For $x \in M$, we define

$$L(M, x) := \inf_{\Gamma_x} \text{length}(\Gamma_x),$$

where the infimum is taken over all graphs Γ_x in M that capture its topology and pass through x.
We also define the function H'' on M as follows.

$$H''(x) := L(M, x) - L(M, h).$$

Finally we define the function H on M as

$$H(x) := min(H'(x), H''(x)),$$

where H' is defined in Definition 6.1.

Definition 6.4. If B is a ball in some closed Riemannian surface M with some contractible boundary components, we fill in every such component of ∂B by an open 2-cell in M and denote by B^+ the union of B with these cells.

Proposition 6.5. *Let (M, h) be a closed Riemannian surface of genus $g \geq 1$ and $x \in M$ with $H(x) < \frac{1}{2}\operatorname{sys}(M, x)$. Then the area of the ball $B(x, R)$ satisfies the inequality*

$$\text{Area } B(x, R) \geq \frac{1}{2}(R - H(x))^2,$$

for every R in the interval $]H(x), \frac{1}{2}\operatorname{sys}(M, x)[$.

Proof. We suppose that $H(x) = H''(x)$ here, since the other case follows from Proposition 6.2. Let $r \in]H(x), \frac{1}{2}\operatorname{sys}(M, x)[$. Notice that since $r < \frac{1}{2}\operatorname{sys}(M, x)$ the ball $B = B(x, r)$ is contractible in M, and so the set $B^+ = B^+(x, r)$ is a topological disk. Let ε be a fixed small positive constant such that $H''(x) + \varepsilon < r$. Fix $\varepsilon' \in (0, \varepsilon)$. Let Γ_x be a graph in M that captures its topology and passes through x of length at most $L(M, x) + \varepsilon'$. Without loss of generality, we claim that we can always suppose that $\Gamma_x \cap B^+(x, r)$ is a tree such that x is the only possible vertex of degree one. Indeed, we delete an edge from each loop of $\Gamma_x \cap B^+(x, r)$. This defines a new graph Γ'. Then we delete every vertex of Γ' of degree one other than the vertex x along with the edge incident to it and we denote by Γ_1 the resulting connected graph. Restart the process. That means we delete every vertex of Γ_1 of degree one other than the vertex x along with the edge incident to it and we denote by Γ_2 the resulting connected graph. By induction, let Γ_k be the last connected subgraph where the only possible vertex of degree one is x. Clearly Γ_k passes through x, captures the topology of M and is of length at most $L(M, x) + \varepsilon'$. So the claim is proved.

Now we claim that either x is of degree at least two or there is at least a vertex of $\Gamma_x \cap B^+$ of degree at least three. Indeed, suppose that x is of

degree one and all the other vertices of $\Gamma_x \cap B^+$ are of degree two. Then $\Gamma_x \cap B^+$ is just a piecewise curve that passes through x and hits ∂B^+ at one point, so its length is greater or equal to r. Thus

$$\text{length}(\Gamma_x) \geq L(M,h) + r.$$

In particular, we have

$$L(M,h) + r \leq L(M,x) + \varepsilon' \leq L(M,x) + \varepsilon.$$

That means

$$r \leq H''(x) + \varepsilon,$$

which is a contradiction.

In both cases above, the graph Γ_x hits the boundary of B^+ in at least two points. Let C be a minimal arc of ∂B^+ that connects the points of $\Gamma_x \cap \partial B^+$. Consider the graph Γ' defined as $(\Gamma_x \setminus (\Gamma_x \cap B^+)) \cup C$. It is clear that Γ' is a connected graph in M that captures its topology, since B^+ is contractible in M. Thus

$$\text{length}(\Gamma') \geq L(M,h).$$

On the other hand, the length of $\Gamma_x \cap B^+$ is at least r. This means that

$$\text{length}(\Gamma_x) \geq \text{length}(\Gamma') + r - \text{length}(C).$$

So

$$L(M,x) + \varepsilon' \geq L(M,h) + r - \text{length}(C).$$

We conclude that for every small positive constant ε', we have

$$H''(x) \geq r - \text{length}(C) - \varepsilon'.$$

Since the length of ∂B^+ is at least the length of the arc C, we have

$$\text{length}(\partial B^+) \geq r - H''(x).$$

By the coarea formula,

$$\begin{aligned} \text{Area}\, B(x,R) &\geq \int_0^R \text{length}(\partial B(x,r))dr \\ &\geq \int_{H''(x)}^R \text{length}(\partial B^+(x,r))dr \\ &= \frac{1}{2}(R - H''(x))^2. \end{aligned}$$

$\square$

7. Existence of ε-regular metrics.

In this section, we define ε-regular metrics and prove their existence. The existence of ε-regular metrics will play a crucial role in controlling the length of minimal graphs on surfaces.

Definition 7.1. Let (M, h) be a closed Riemannian surface. The metric h is called ε-regular if for all the points x in M, $H(x) \leq \varepsilon$.

Lemma 7.2. *Let (M_0, h_0) be a closed Riemannian surface. Then for every $\varepsilon > 0$, there exists a Riemannian metric $\bar{h}$ on M_0 conformal to h_0 such that*

(1) $\mathrm{Area}(M_0, \bar{h}) \leq \mathrm{Area}(M_0, h_0)$;
(2) $\bar{h}$ *is ε-regular;*
(3) $L(M_0, \bar{h}) = L(M_0, h_0)$*;*
(4) $\mathrm{sys}(M_0, \bar{h}) = \mathrm{sys}(M_0, h_0)$.

Proof. Take a point x_0 in M_0 where $H(x_0) = H_{h_0}(x_0) > \varepsilon$ and denote by M_1 the space M_0/B^+ obtained by collapsing $B^+ = B^+(x_0, \varepsilon)$ to x_0. Let $p_0 : M_0 \to M_1$ be the (non-expanding) canonical projection and h_1 be the metric induced by h on M_1. The Riemannian surface (M_1, h_1) clearly satisfies (1). If h_1 is not ε-regular, we apply the same process. By induction we construct a sequence of :

- balls $B_i^+ = B^+(x_i, \varepsilon)$ in M_i, where x_i is a point with $H_{h_i}(x_i) > \varepsilon$.
- Riemannian surfaces (M_i, h_i) where $M_i = M_{i-1}/B_{i-1}$ and h_i is the metric induced by h_{i-1} on M_i.
- non-expanding canonical projections $p_i : M_i \to M_{i+1}$.

This process stops when we get an ε-regular metric.

Now, we argue exactly as [14, Lemma 4.2] to prove that this process stops after finitely many steps. Let $B_1^i, \ldots, B_{N_i}^i$ be a maximal system of disjoint balls of radius $r/3$ in M_i. Since p_{i-1} is non-expanding, the preimage $p_{i-1}^{-1}(B_k^i)$ of B_k^i contains a ball of radius $r/3$ in M_{i-1}. Furthermore, the preimage $p_{i-1}^{-1}(x_i)$ of x_i contains a ball B_{i-1} of radius r in M_{i-1}. Thus, two balls of radius $r/3$ lie in the preimage of x_i under p_{i-1}. It is then possible to construct a system of $N_i + 1$ disjoint disks of radius $r/3$ in M_{i-1}. Thus, $N_{i-1} \geq N_i + 1$ where N_i is the maximal number of disjoint balls of radius $r/3$ in M_i. Therefore, the process stops after N steps with $N \leq N_0$. Denote by h_N the metric where this process stops. Clearly h_N satisfies (1) and (2). To see that h_N satisfies (3) and (4), let Γ be a minimal graph in M_0 and α be a systolic loop in M. For every point x in the ε-neighborhood N_Γ of Γ, we have $H(x) \leq \varepsilon$. Indeed, let c be a minimizing curve from Γ to x. The graph $\Gamma \cup c$ captures the topology of M_0 and passes through x. So $H''(x) \leq \mathrm{length}(\Gamma \cup c) - L(M, h) \leq \varepsilon$. That means that the balls we collapsed through the whole process do not intersect Γ. Therefore, the

metric h_N satisfies (3). A similar argument holds for α. So the metric h_N also satisfies (4).

$\square$

8. Construction of short minimal graphs on surfaces

In this section, we combine Lemma 7.2 and the construction of [2, p. 46] to construct a minimal graph with controlled length on a given Riemannian surface.

Proposition 8.1. *Let (M,h) be a closed orientable Riemannian surface of genus $g \geq 2$. Suppose that*

- $\mathrm{Area}(M,h) \leq \frac{1}{2^{12}}(2g-1)$;
- $\mathrm{sys}(M,h) \geq \frac{1}{2}$.

Then

$$L(M,h) \leq \frac{1}{2}(2g-1).$$

Proof. Fix $r_0 = \frac{1}{2^5}$. By Lemma 7.2 (choose ε small enough) and Proposition 6.5, there exists a conformal Riemannian metric $\bar{h}$ on M that satisfies

(1) The area of every disk of $(M,\bar{h})$ of radius r_0 is at least $\frac{1}{4}{r_0}^2$;
(2) $\mathrm{Area}(M,\bar{h}) \leq \mathrm{Area}(M,h)$;
(3) $L(M,\bar{h}) = L(M,h)$;
(4) $\mathrm{sys}(M,\bar{h}) = \mathrm{sys}(M,h)$
(5) $\bar{h}$ is ε-regular.

So it is sufficient to prove that

$$L(M,\bar{h}) \leq \frac{1}{2}(2g-1).$$

Let $\{B_i\}_{i\in I}$ be a maximal system of disjoint balls of radius r_0 in $(M,\bar{h})$. Since the area of each ball B_i is at least $\frac{1}{4}r_0^2$, then

$$\frac{1}{4}|I|r_0^2 \leq \mathrm{Area}(M,\bar{h}),$$

that is,

$$|I| \leq 2^{12}\,\mathrm{Area}(M,\bar{h}). \tag{8.1}$$

As this system is maximal, the balls $2B_i$ of radius $2r_0$ with the same centers p_i as B_i cover M.

Let ε be a small positive constant that satisfies

$$4r_0 + 2\varepsilon < \frac{1}{4} \leq \frac{\mathrm{sys}(M,\bar{h})}{2},$$

and denote by $2B_i + \varepsilon$ the balls centered at p_i with radius $2r_0 + \varepsilon$. We construct an abstract graph Γ as follows. Let $\{w_i\}_{i\in I}$ be a set of vertices corresponding to $\{p_i\}_{i\in I}$. Two vertices w_i and $w_{i'}$ of Γ are linked by an edge if and only if the balls $2B_i + \varepsilon$ and $2B_{i'} + \varepsilon$ intersect each other. Define a metric on Γ such that the length of each edge is $\frac{1}{4}$ and let $\varphi : \Gamma \to M$ be the

map that sends each edge of Γ with endpoints w_i and $w_{i'}$ to a minimizing geodesic joining p_i and $p_{i'}$. Since $\mathrm{dist}(p_i, p_{i'}) \leq 4r_0 + 2\varepsilon < \frac{1}{4}$, the map φ is distance nonincreasing.

Claim. The map $\varphi_\star : \pi_1(\Gamma) \to \pi_1(M)$ induced by φ between the fundamental groups is an epimorphism. In particular, it induces an epimorphism in real homology.

We argue exactly as [2, Lemma 2.10]. Consider a geodesic loop σ of M. Divide the loop σ into segments $\sigma_1, \dots, \sigma_n$ of length at most ε. Denote by x_k and x_{k+1} the endpoints of σ_k with the convention $x_{n+1} = x_1$. Recall that the balls $2B_i$ cover the surface M. So every point x_k is at distance at most $2r_0$ from a point v_k among the centers p_i. Let β_k be the loop

$$\sigma_k \cup C_{x_{k+1}v_{k+1}} \cup C_{v_{k+1},v_k} \cup C_{v_k,x_k},$$

where C_{ab} denotes a minimizing geodesic joining a to b. We have that

$$\mathrm{length}(\beta_k) \leq 2\,(4r_0 + \varepsilon) < \mathrm{sys}(M, \bar{h}).$$

That means that the loops β_k are contractible. We conclude that the loop σ is homotopic to a piecewise geodesic loop $\sigma' = (v_1, \dots, v_n)$.
The distance between the centers $v_k = p_{i_k}$ and $v_{k+1} = p_{i_{k+1}}$ is less than or equal to $4r_0 + \varepsilon$. So the vertices w_{i_k} and $w_{i_{k+1}}$ of Γ corresponding to the vertices p_{i_k} and $p_{i_{k+1}}$ are connected by an edge. The union of these edges forms a loop $(w_{i_1}, \dots, w_{i_n})$ in Γ whose image by the map φ is σ'. Since σ' is homotopic to σ, the claim is proved.

Now we consider a connected subgraph Γ' of Γ with a minimal number of edges such that the restriction of φ to Γ' still induces an epimorphism in real homology.

We claim that the epimorphism $\varphi_* : H_1(\Gamma'; \mathbb{R}) \to H_1(M; \mathbb{R})$ is an isomorphism. Indeed, if φ_* is not an isomorphism then arguing as in Proposition 5.2 we can remove at least one edge of Γ' such that φ_* is still an epimorphism, which is impossible by the definition of Γ'.
We denote by v, e, b and b' respectively the number of vertices of Γ, the number of edges of Γ, the first Betti number of Γ and the first Betti number of Γ'. At least $b - b'$ edges were removed from Γ to obtain Γ'. As $b' = 2g$, we derive

$$\begin{aligned} \mathrm{length}(\Gamma') &\leq \mathrm{length}(\Gamma) - (b - b')\,\frac{1}{4} \\ &\leq (e - b + 2g)\,\frac{1}{4} \\ &\leq (v - 1 + 2g)\,\frac{1}{4}. \end{aligned} \tag{8.2}$$

Recall that $\mathrm{Area}(M,\bar{h}) \leq \frac{1}{2^{12}}(2g-1)$. So

$$v = |I| \leq 2g-1.$$

Combining this with (6.2), we get

$$\mathrm{length}(\Gamma') \leq \frac{1}{2}(2g-1).$$

Since φ is distance non-increasing then

$$\mathrm{length}(\varphi(\Gamma')) \leq \mathrm{length}(\Gamma').$$

The image by φ of two edges of Γ' may intersect. If it is the case then the intersection point should be considered as a vertex of the graph $\varphi(\Gamma')$. Thus the set of vertices of $\varphi(\Gamma')$ may be bigger than the set of vertices of Γ'.

Finally let j be the inclusion map $j : \varphi(\Gamma') \hookrightarrow M$. Clearly the map $j_* : H_1(\varphi(\Gamma'); \mathbb{R}) \to H_1(M; \mathbb{R})$ is an epimorphism. So $\varphi(\Gamma')$ is a graph in M that captures its topology. Thus

$$L(M,\bar{h}) \leq \mathrm{length}(\varphi(\Gamma')) \leq \frac{1}{2}(2g-1).$$

$\square$

9. Proofs of Theorem A and Theorem B.

In this section, we prove Theorem A and Theorem B. But before doing that we examine how the function V changes with scaling. Let (M^n, h) be a closed n-dimensional Riemannian manifold and $h' = \lambda^2 h$ with $\lambda > 0$ then

- $\mathrm{Vol}(M, h') = \lambda^n \,\mathrm{Vol}(M, hyp)$;
- $V_{(\widetilde{M},\widetilde{h'})}(\lambda R) = \lambda^n V_{(\widetilde{M},\widetilde{h})}(R)$.

The expression (1.1) of $V_{\mathbb{H}^2}$ immediately leads to the following lemma.

Lemma 9.1. *Let a be a positive constant. There exists a constant $c = c(a)$ such that for all $R \geq 0$,*

$$aV_{\mathbb{H}^2}(R) \geq V_{\mathbb{H}^2}(Rc).$$

In light of Lemma 9.1, the proof of Theorem B amounts to proving the following result.

Theorem 9.2. *Let (M, hyp) be a closed hyperbolic surface of genus g and h be another Riemannian metric on M with*

$$\mathrm{Area}(M,h) \leq \frac{1}{2^{13}\pi}\,\mathrm{Area}(M, hyp).$$

Then, for any radius $R \geq 0$,

$$V_{(\widetilde{M},\widetilde{h})}(R) \geq \frac{1}{4\pi \ln 2} V_{\mathbb{H}^2}(R \ln 2).$$

In particular, there exists a constant c such that

$$V_{(\widetilde{M},\widetilde{h})}(R) \gtrsim c\, 2^R,$$

when R tends to infinity.

Proof. Let $R > 0$. First, we consider the special case when M is oriented and

$$\mathrm{sys}(M,h) \geq max\{2R, 1/2\}.$$

In this case,

$$V_{(M,h)}(R) = V_{(\widetilde{M},\widetilde{h})}(R).$$

Let Γ be a minimal graph which captures the topology of (M,h) (*cf.* Definition 5.5). Denote by $b = 2g$ the first Betti number of Γ. We have

$$\mathrm{Area}(M,h) \leq \frac{1}{2^{13}\pi}\,\mathrm{Area}(M,hyp) \leq \frac{1}{2^{12}}(2g-1).$$

So by Proposition 8.1 and the relation (2.3), we have

$$\mathrm{length}(\Gamma) \leq \frac{1}{2}(b-1) = \frac{1}{6}\,\mathrm{length}(\Gamma_b, h_b). \tag{9.1}$$

Let v be any vertex of Γ. Denote by $B(v,R)$ the ball in (M,h) centered at v with radius R. We claim that for all $r \in (0,R)$

$$\mathrm{length}(\partial B^+(v,r)) \geq \mathrm{length}(\Gamma \cap B^+(v,r)), \tag{9.2}$$

where $B^+(v,r)$ is defined in Definition 6.4.
We argue as in Proposition 6.5. Suppose the opposite and replace $\Gamma \cap B^+(v,r)$ by a minimal arc of $\partial B^+(v,r)$ that links the points of $\Gamma \cap \partial B^+(v,r)$. Since $B^+(v,r)$ is contractible, the new graph captures the topology of M and is shorter than Γ which contradicts the definition of Γ.

Let $B_{(\Gamma,h)}(v,r)$ be the ball centered at v of radius R in the metric graph (Γ,h). Since the ball $B_{(\Gamma,h)}(v,r)$ is contained in $\Gamma \cap B^+(v,r)$, we have

$$\mathrm{length}(\Gamma \cap B^+(v,r)) \geq \mathrm{length}(B_{(\Gamma,h)}(v,r)). \tag{9.3}$$

Let $\tilde{v}$ be a lift of v in $\widetilde{\Gamma}$. Since $\mathrm{sys}(M,h) \leq \mathrm{sys}(\Gamma,h)$, we have for $r \leq \frac{1}{2}\,\mathrm{sys}(M,h)$

$$\mathrm{length}(B_{(\Gamma,h)}(v,r)) = \mathrm{length}(B_{(\widetilde{\Gamma},\widetilde{h})}(\tilde{v},r)). \tag{9.4}$$

By Theorem C (take $\lambda = \frac{1}{6}$) and the bound (9.1), there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that

$$\mathrm{length}(B_{(\widetilde{\Gamma},\widetilde{h})}(\tilde{u},r)) \geq \frac{1}{2}V'_{(\widetilde{\Gamma}_{2g},\widetilde{h}_{2g})}(r).$$

Denote by u the image of $\tilde{u}$ by the covering map. By (9.2), (9.3), (9.4) and (2.4), we obtain

$$\begin{aligned}\mathrm{length}(\partial B^+(u,r)) &\geq \frac{1}{2}V'_{(\widetilde{\Gamma}_{2g},\widetilde{h}_{2g})}(r)\\ &\geq \frac{1}{2}\sinh(r\ln 2).\end{aligned}$$

By the coarea formula,

$$\begin{aligned}\operatorname{Area}(B(u,R)) &\geq \frac{1}{2}\int_0^R \sinh(r\ln 2)dr \\ &= \frac{1}{2\ln 2}(\cosh(R\ln 2)-1). \\ &= \frac{1}{4\pi\ln 2}V_{\mathbb{H}^2}(R\ln 2).\end{aligned}$$

Next, we consider the general case with no restriction on the systole and the orientability of M. Since M admits a hyperbolic metric, the fundamental group of M is residually finite (see [12]). Therefore, we can choose a finite cover $(\bar{M},\bar{h})$ such that $\bar{M}$ is orientable and

$$\operatorname{sys}(\bar{M},\bar{h}) \geq max\{2R, 1/2\}.$$

Let $\bar{hyp}$ be the pullback of the hyperbolic metric on M to $\bar{M}$.
Now, if the covering $\pi : \bar{M} \to M$ has degree d, then $\operatorname{Area}(\bar{M},\bar{h}) = d\operatorname{Area}(M,h)$ and $\operatorname{Area}(\bar{M},\bar{hyp}) = d\operatorname{Area}(M,hyp)$. So

$$\operatorname{Area}(\bar{M},\bar{h}) \leq \frac{1}{2^{13}\pi}\operatorname{Area}(\bar{M},\bar{hyp}).$$

Finally, since the universal cover of $(\bar{M},\bar{h})$ agrees with the universal cover of (M,h), we can conclude by the first case. $\square$

Now we prove Theorem A.

Proof of Theorem A. Let (M,hyp) be a closed hyperbolic Riemannian surface of genus g. Let δ be a small positive constant and h another metric on M with $\operatorname{Area}(M,h) \leq \delta\operatorname{Area}(M,hyp)$. We will show that if we take δ small enough (independently from the metric h) then for any radius $R \geq 1$,

$$V_{(\widetilde{M},\widetilde{h})}(R) \geq V_{\mathbb{H}^2}(R).$$

Indeed, let $h' = \lambda^2 h$ where λ is a positive constant such that

$$\operatorname{Area}(M,h') = \frac{1}{2^{13}\pi}\operatorname{Area}(M,hyp).$$

By Theorem 9.2, we have that for any radius $R \geq 0$,

$$V_{(\widetilde{M},\widetilde{h}')}(R) \geq \frac{1}{4\pi\ln 2}V_{\mathbb{H}^2}(R\ln 2).$$

Recall that

$$\operatorname{Area}(M,h') = \lambda^2\operatorname{Area}(M,h) \leq \lambda^2\delta\operatorname{Area}(M,hyp).$$

So

$$\lambda^2 \geq \frac{1}{2^{13}\pi\delta}.$$

On the other hand, we have

$$V_{(\widetilde{M},\widetilde{h}')}(\lambda R) = \lambda^2 V_{(\widetilde{M},\widetilde{h})}(R).$$

So

$$V_{(\widetilde{M},\widetilde{h})}(R) \geq \frac{1}{4\pi\lambda^2 \ln 2} V_{\mathbb{H}^2}(\lambda R \ln 2).$$

Now we choose λ large enough so that for all $R \geq 1$ we have

$$\frac{1}{4\pi\lambda^2 \ln 2} V_{\mathbb{H}^2}(\lambda R \ln 2) \geq V_{\mathbb{H}^2}(R).$$

To see that such a λ exists notice that for $R \geq 1$ we have

$$\frac{1}{4\pi\lambda^2 \ln 2} V_{\mathbb{H}^2}(\lambda R \ln 2) \geq \frac{1}{8\pi\lambda^2 \ln 2}(e^{\frac{\lambda \ln 2}{2}} e^{\frac{\lambda R \ln 2}{2}} - 2).$$

When λ tends to infinity, the number $\frac{1}{8\pi\lambda^2 \ln 2} e^{\frac{\lambda \ln 2}{2}}$ tends to infinity and so

$$\frac{1}{8\pi\lambda^2 \ln 2}(e^{\frac{\lambda \ln 2}{2}} e^{\frac{\lambda R \ln 2}{2}} - 2) \gg V_{\mathbb{H}^n}(R).$$

Recall that to get λ large enough it suffices to choose δ small enough.
Finally, we would like to point out that when R tends to zero we cannot find a λ such that

$$\frac{1}{4\pi\lambda^2 \ln 2} V_{\mathbb{H}^2}(\lambda R \ln 2) \geq V_{\mathbb{H}^2}(R).$$

□

References

[1] Besson, G.; Courtois, G.; Gallot, S., Volumes, entropies et rigidités des espaces localement symétriques de courbure strictement négative, Geom. Funct. Anal. 5 (1995), no. 5, 731-799.

[2] Balacheff, F.; Parlier H.; Sabourau S., Short loops decompositions of surfaces and the geometry of Jacobians, Geom. Funct. Anal. 22 (2012), no. 1, 37-73.

[3] Bollobás, B; Szemerédi, E., Girth of sparse graphs, J. Graph Theory 39 (2002), 194-200.

[4] Diestel. Reinhard., Graph theory, Springer, Heidelberg, 2010.

[5] Gromov, M., Filling Riemannian manifolds, J. Differential Geom. 18 (1983) no. 1, 1-147.

[6] Gromov, M., Large Riemannian manifolds, in *Curvature and topology of Riemannian manifolds* (Katata, 1985), 108-121, Lecture Notes in Math. 1201, Springer, Berlin, 1986.

[7] Gromov, M., Metric Structures for Riemannian and Non-Riemannian Spaces, Progr. Math. 152, Birkhauser, Boston, MA, 1999. MR 1699320. Zbl 0953.53002.

[8] Guth, L., Volumes of balls in large Riemannian manifolds. Ann. of Math. (2) 173 (2011), no. 1, 51-76.

[9] Katok, A. Entropy and closed geodesics, Ergod. Th. Dynam. Sys. 2 (1983) 339-365.

[10] Kapovich, I; Nagnibeda, T; The Patterson-Sullivan embedding and minimal volume entropy for outer space. Geom. Funct. Anal. 17 (2007), no. 4, 1201-1236.

[11] Lim, S., Minimal volume entropy for graphs. Trans. Amer. Math. Soc. 360 (2008), no. 10, 5089-5100.

[12] Malcev, A., On isomorphic matrix representations of infinite groups, Rec. Math. [Mat. Sbornik] 8 (50) (1940), 405-422.

[13] Manning, A., Topological entropy for geodesic flows. Ann. of Math. 110 (1979), no. 2, 567-573.

[14] Rudyak, Yuli B.; Sabourau, S., Systolic invariants of groups and 2-complexes via Grushko decomposition. Ann. Inst. Fourier (Grenoble) 58 (2008), no. 3, 777-800.

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